Appendix A

Moments of the Probability Density Functions

The function $g(x) = x^n$ leads to the general moments of the random variable [144 p64]:

$$\overline{X^n} = E[X^n] = \int_{-\infty}^{\infty} x^n f(x)dx$$

From the probability distribution functions derived in chapter 5.5 the calculation of the relevant moments of the distributions are shown in this appendix.

A.1 Rayleigh pdf

The mean of the Rayleigh distribution is found from:

$$\bar{Z} = \int_0^{\infty} z \cdot m_Z(z)dz = \int_0^{\infty} \frac{z^2}{\sigma_{ft}^2} e^{\frac{-z^2}{2\sigma_{ft}^2}} dz = \sqrt{\frac{\pi}{2}} \cdot \sigma_{ft}$$

And the mean square value:

$$\bar{Z}^2 = \int_0^{\infty} z^2 \cdot m_Z(z)dz = \int_0^{\infty} \frac{z^3}{\sigma_{ft}^2} e^{\frac{-z^2}{2\sigma_{ft}^2}} dz = 2\sigma_{ft}^2$$

Subtraction of the square of the mean from the mean square value will result in the variance value:

$$\sigma_Z^2 = \bar{Z}^2 - \left(\bar{Z}\right)^2 = \int_0^{\infty} \frac{z^3}{\sigma_{ft}^2} e^{\frac{-z^2}{2\sigma_{ft}^2}} dz - \left[\int_0^{\infty} \frac{z^2}{\sigma_{ft}^2} e^{\frac{-z^2}{2\sigma_{ft}^2}} dz\right]^2 = \left(2 - \frac{\pi}{2}\right) \cdot \sigma_{ft}^2 = 0.429\sigma_{ft}^2$$

The Rayleigh pdf is asymmetrical resulting in different values for the mean and the mode (point of maximum likelihood). The pdf is also uni-modal, so in order to find the mode the pdf can be integrated producing a result that will be zero at the turning point or peak in the pdf :

$$\frac{d}{dz}\frac{z}{\sigma_{ft}^2}\mathrm{e}^{\frac{-z^2}{2\sigma_{ft}^2}} = -\mathrm{e}^{\left(\frac{-z^2}{2\sigma_{ft}^2}\cdot\frac{\sigma^2+z^2}{\sigma_{ft}^4}\right)}$$

Equating z to zero in this equation shows that the mode is equal to the standard deviation: $\hat{z} = \sigma_{ft}$

A.2 Power pdf

The mean of the power distribution can be found from:

$$\bar{Q} = \int_0^\infty q \cdot p_Q(q)dq = \int_0^\infty \frac{q}{2\sigma_{ft}^2}\mathrm{e}^{\frac{-q}{2\sigma_{ft}^2}}dq = 2\sigma_{ft}^2$$

and the mean square value:

$$\bar{Q}^2 = \int_0^\infty q^2 \cdot p_Q(q)dq = \int_0^\infty \frac{q^2}{2\sigma_{ft}^2}\mathrm{e}^{\frac{-q}{2\sigma_{ft}^2}}dq = 8\sigma_{ft}^4$$

Subtraction of the square of the mean from the mean square value will result in the variance value:

$$\sigma_Q^2 = \bar{Q}^2 - \left(\bar{Q}\right)^2 = \int_0^\infty \frac{q^2}{2\sigma_{ft}^2}\mathrm{e}^{\frac{-q}{2\sigma_{ft}^2}}dq - \left(\int_0^\infty \frac{q}{2\sigma_{ft}^2}\mathrm{e}^{\frac{-q}{2\sigma_{ft}^2}}dq\right)^2 = 8\sigma_{ft}^4 - 4\sigma_{ft}^4 = 4\sigma_{ft}^4$$

A.3 Logarithm of the Power pdf

The mean value of the logarithm of the power pdf can be found from:

$$\bar{R} = \int_{-\infty}^\infty r \cdot l_R(r)dr = \int_{-\infty}^\infty \frac{r\cdot \mathrm{e}^r}{2\sigma_{ft}^2}\mathrm{e}^{\frac{-\mathrm{e}^r}{2\sigma_{ft}^2}}dr$$

and the mean square value:

$$\bar{R}^2 = \int_{-\infty}^\infty r^2 \cdot l_R(r)dr = \int_{-\infty}^\infty \frac{r^2\cdot \mathrm{e}^r}{2\sigma_{ft}^2}\mathrm{e}^{\frac{-\mathrm{e}^r}{2\sigma_{ft}^2}}dr$$

Subtraction of the square of the mean from the mean square value will result in the variance value:

$$\sigma_R^2 = \overline{R}^2 - \left(\overline{R}\right)^2 = \int_{-\infty}^{\infty} \frac{r^2 \cdot \mathrm{e}^r}{2\sigma_{ft}^2} \mathrm{e}^{\frac{-\mathrm{e}^r}{2\sigma_{ft}^2}} dr \; - \left[\int_{-\infty}^{\infty} \frac{r \cdot \mathrm{e}^r}{2\sigma_{ft}^2} \mathrm{e}^{\frac{-\mathrm{e}^r}{2\sigma_{ft}^2}} dr \right]^2 = \frac{\pi^2}{6} = 1.645$$

The log of power pdf is asymmetrical producing different values for the mean and the mode (point of maximum likelihood). The pdf is also uni-modal, so in order to find the mode the pdf will be integrated producing a result that will be zero at the turning point or peak in the pdf :

$$\frac{d}{dr} \frac{\mathrm{e}^r}{2\sigma_{ft}^2} \mathrm{e}^{\frac{-\mathrm{e}^r}{2\sigma_{ft}^2}} = \frac{\left[2\sigma_{ft}^2 \cdot e^{\frac{2r\sigma_{ft}^2 - e^r}{2\sigma_{ft}^2}} \right] - \left[e^{\frac{4r\sigma_{ft}^2 - e^r}{2\sigma_{ft}^2}} \right]}{4\sigma^2}$$

Equating *r* in this equation to zero produces the mode:

$$\hat{R} = \log_e(2\sigma_{ft}^2)$$

The mean is found to have a value $\Upsilon = 0.577$ lower than the mode, providing a closed form solution for the mean:

$$\overline{R} = \log_e(2\sigma_{ft}^2) - \Upsilon$$

A.4 Reverse Difference Transform pdf

The mean value is given by:

$$\overline{A} = \int_{-\infty}^{\infty} \frac{a\left(\sigma_{\omega 1}^2 \cdot \sigma_{\omega 2}^2 \cdot e^a\right)}{\left(\sigma_{\omega 2}^2 + \sigma_{\omega 1}^2 \cdot e^a\right)^2} da$$

After integrating becomes:

$$\overline{A} = 2\log_e\left(\frac{\sigma_{\omega 2}}{\sigma_{\omega 1}}\right)$$

The mean square value is given by:

$$\overline{A}^2 = \int_{-\infty}^{\infty} \frac{a^2\left(\sigma_{\omega 1}^2 \cdot \sigma_{\omega 2}^2 \cdot e^a\right)}{\left(\sigma_{\omega 2}^2 + \sigma_{\omega 1}^2 \cdot e^a\right)^2} da$$

After integrating becomes:

$$\overline{A}^2 = 4\log_e\left(\sigma_{\omega 1}\right)^2 - 8\log_e\left(\sigma_{\omega 1}\right) \cdot \log_e\left(\sigma_{\omega 2}\right) + 4\log_e\left(\sigma_{\omega 2}\right)^2 + \frac{\pi^2}{3}$$

The variance is obtained by subtracting the square of the mean from the mean square values:

$$\sigma_A^2 = \overline{A}^2 - \left(\overline{A}\right)^2 =$$

$$\left[4\log_e\left(\sigma_{\omega1}\right)^2 - 8\log_e\left(\sigma_{\omega1}\right)\cdot\log_e\left(\sigma_{\omega2}\right) + 4\log_e\left(\sigma_{\omega2}\right)^2 + \frac{\pi^2}{3}\right] - \left[2\log_e\left(\frac{\sigma_{\omega2}}{\sigma_{\omega1}}\right)\right]^2 = \frac{\pi^2}{3} = 3.29$$

The skewness, γ or 3rd central moment of a distribution is defined as the ratio of the 3rd moment u^3 about the mean $E(a)$ to the 3rd power of the standard deviation [178] and given by:

$$\gamma = \frac{\mu^3}{\sigma_A^3} = \frac{E\left[A - E(A)\right]^3}{\left\{E\left[A - E(A)\right]^2\right\}^{\frac{3}{2}}}$$

The 3rd moment can be given by:

$$\mu^3 = E\left(a - \overline{A}\right)^3$$

The skewness value is found to be a constant for the RDT distribution:

$$\gamma_A = \int_{-\infty}^{\infty} \frac{\left(a - \overline{A}\right)^3 \cdot \left(\sigma_{\omega1}^2 \cdot \sigma_{\omega2}^2 \cdot e^a\right)}{\sigma_A^3 \cdot \left(\sigma_{\omega2}^2 + \sigma_{\omega1}^2 \cdot e^a\right)^2} da = 0$$

The kurtosis, K or 4th central moment of a distribution is defined as the ratio of the 4th moment μ^4 about the mean $E(a)$ to the 4th power of the standard deviation [178].

$$\mathrm{K} = \frac{\mu^4}{\sigma_A^4} = \frac{E\left[A - E(A)\right]^4}{\left\{E\left[A - E(A)\right]^2\right\}^2}$$

The 4th moment is given by:

$$\mu^4 = E\left(a - \overline{A}\right)^4$$

The kurtosis value is found to be a constant for the RDT distribution:

$$\mathrm{K}_A = \int_{-\infty}^{\infty} \frac{\left(a - \overline{A}\right)^4 \cdot \left(\sigma_{\omega1}^2 \cdot \sigma_{\omega2}^2 \cdot e^a\right)}{\sigma_A^4 \cdot \left(\sigma_{\omega2}^2 + \sigma_{\omega1}^2 \cdot e^a\right)^2} da = 4.2$$